# Powers & Roots of Complex Numbers

$i = \sqrt{-1}$ The imaginary unit

$i^2 = -1$

$i^3 = i^2 \cdot i = -i$

$\boxed{\pm i, \pm 1}$

$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$

$i^{260} = (i^4)^{65} = 1^{65} = 1$

$i^{995} = i^{992} \cdot i^3 = i^3 = -i$

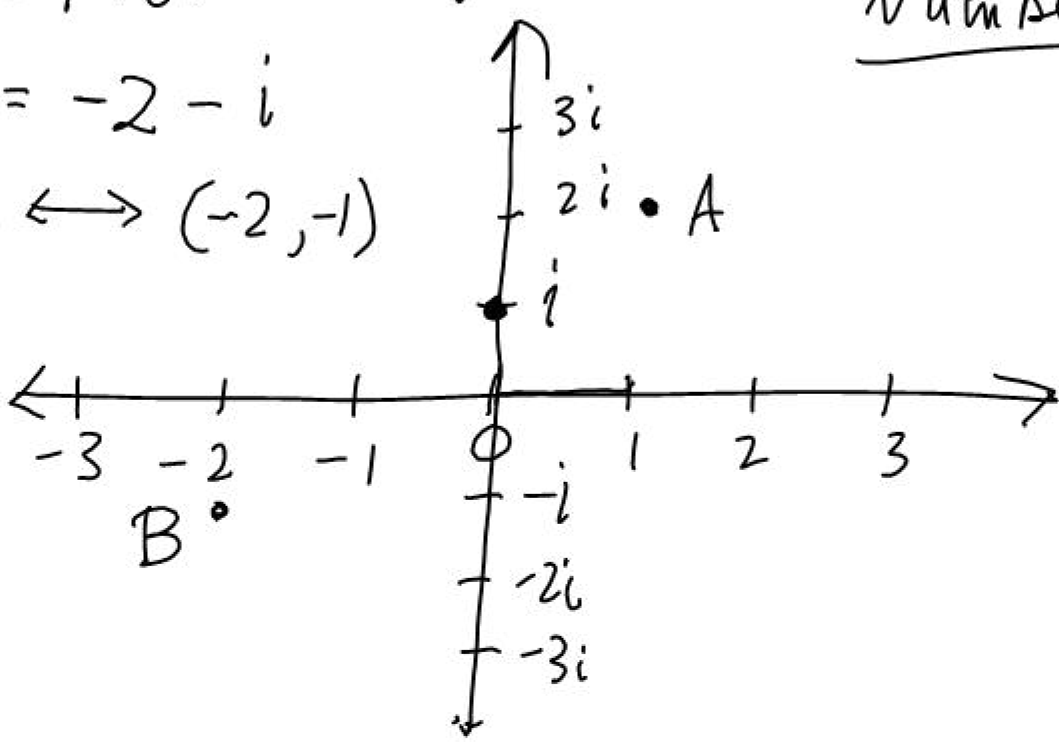
multiple of 4

$A = 1 + 2i$

$B = -2 - i$

$B \longleftrightarrow (-2, -1)$

Number plane



# Polar Coordinates for Complex Numbers

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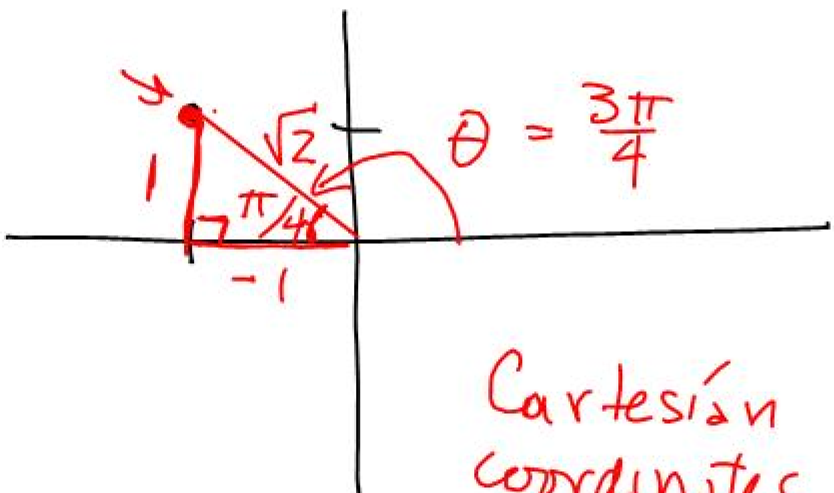
Ex. Find the polar coordinates for $-1+i$ $\qquad\qquad (-1+i)^5$




Cartesian coordinates $\qquad$ polar coordinates

$$-1+i \longleftrightarrow (-1,\ 1) \longleftrightarrow \left(\sqrt{2},\ \tfrac{3\pi}{4}\right)$$

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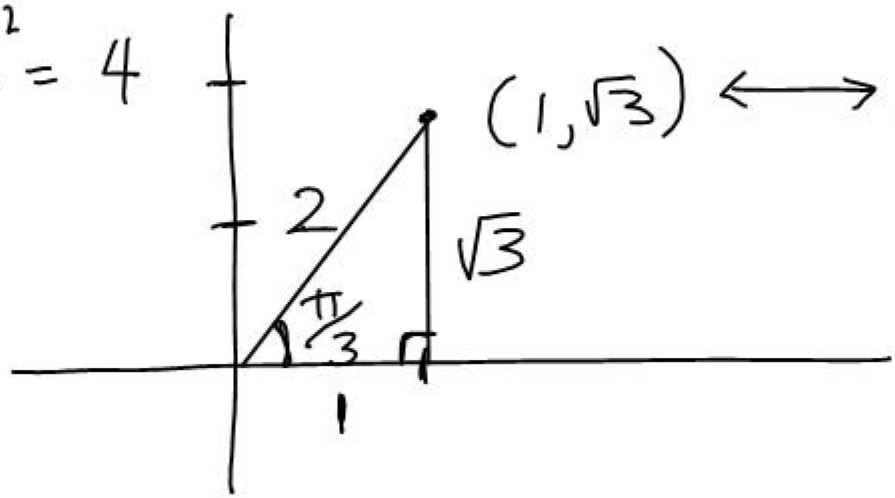
Find the polar coordinates for $1 + i\sqrt{3}$

$1^2 + \sqrt{3}^2 = 4$




$$(1,\sqrt{3}) \longleftrightarrow \left(2,\ \tfrac{\pi}{3}\right)$$

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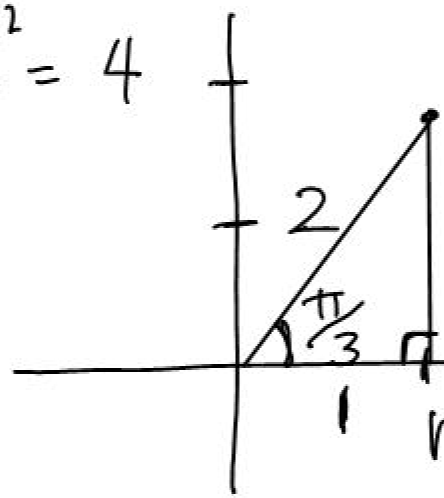
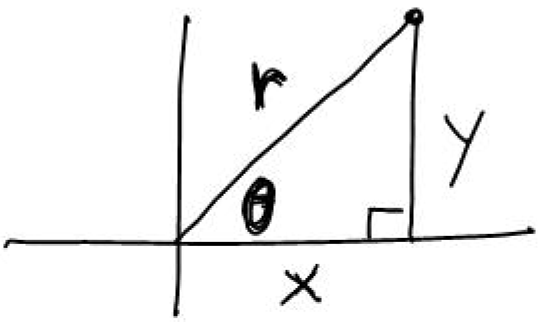



$(1, \sqrt{3}) \longleftrightarrow (2, \frac{\pi}{3})$

$r\cos\theta = 2\cos\frac{\pi}{3}$

$= 2\left(\frac{1}{2}\right) = 1$ ✓

$r\sin\theta = 2\sin\frac{\pi}{3} = \cancel{2\left(\frac{\sqrt{3}}{2}\right)} = \sqrt{3}$ ✓




Given r and θ,

find x:

$\frac{x}{r} = \cos\theta$

$\boxed{x = r\cos\theta}$

Find y:

$\frac{y}{r} = \sin\theta$

$\boxed{y = r \cdot \sin\theta}$

$r$ = modulus

$\theta$ = argument

$x + i \cdot y = r \cdot \cos\theta + i \cdot r \cdot \sin\theta$

$= \underline{\underline{r(\cos\theta + i \cdot \sin\theta)}}$

Modulus - Argument form :

$$r(\cos\theta + i \cdot \sin\theta) = r \cdot \text{cis}\,\theta$$

Ex. The modulus-argument form for $-1+i$ is :

$$\sqrt{2}\left(\cos\frac{3\pi}{4} + i \cdot \sin\frac{3\pi}{4}\right) = \sqrt{2}\,\text{cis}\,\frac{3\pi}{4}$$

check $\sqrt{2}\left(-\frac{1}{\sqrt{2}} + i \cdot \frac{1}{\sqrt{2}}\right)$

$= -1 + i \cdot 1$

$= -1 + i$ ✓

# DeMoivre's Theorem

$$\boxed{(r \cdot \text{cis}\,\theta)^n = r^n \cdot \text{cis}(n\theta)}$$

Ex. $(1 + i\sqrt{3})^{10}$

$$= \left(2\,\text{cis}\,\frac{\pi}{3}\right)^{10} = 2^{10} \cdot \text{cis}\left(\frac{10\pi}{3}\right)$$

III coterminal

$$= 1024\,\text{cis}\left(\frac{4\pi}{3}\right)$$

$$= 1024\left(-\frac{1}{2} - i \cdot \frac{\sqrt{3}}{2}\right)$$

$$= \underline{-512 - 512\sqrt{3}\,i}$$

Ex $(-1+i)^5 = \left(\sqrt{2}\,\text{cis}\,\frac{3\pi}{4}\right)^5$

$2 \cdot 2 \cdot \sqrt{2}$

$\cancel{\sqrt{2}\sqrt{2}}\;\cancel{\sqrt{2}\sqrt{2}}\;\sqrt{2}$

$= \sqrt{2}^{\,5}\,\text{cis}\left(\frac{15\pi}{4}\right)$

$= 4\sqrt{2}\,\text{cis}\left(\frac{7\pi}{4}\right)$

$= 4\sqrt{2}\left(\frac{1}{\sqrt{2}} - i\cdot\frac{1}{\sqrt{2}}\right)$

$= \boxed{4 - 4i}$

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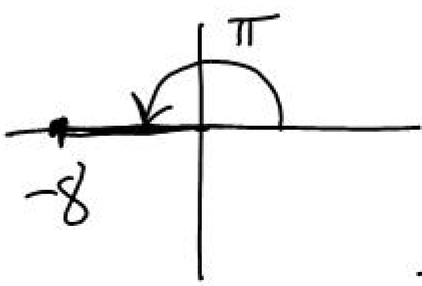
Ex. Find the $\underline{3}$ cube roots of $-8$.

$-8+0i = 8\,\text{cis}\,\pi$

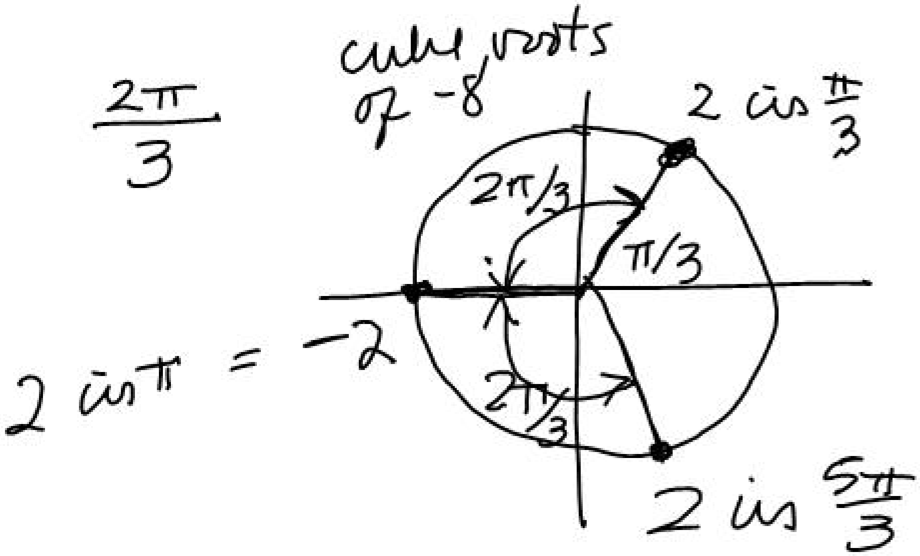



$(-8)^{\frac{1}{3}} = (8\,\text{cis}\,\pi)^{\frac{1}{3}}$

$= 8^{\frac{1}{3}} \cdot \text{cis}\left(\frac{\pi}{3}\right) = 2\,\text{cis}\,\frac{\pi}{3}$

$= 2\left(\frac{1}{2} + i\cdot\frac{\sqrt{3}}{2}\right) = \underline{1 + i\sqrt{3}}$

$\frac{2\pi}{3}$




roots are equally spaced on a circle

$$\boxed{\begin{array}{l} \bullet\ -2 \\ \bullet\ 2 \operatorname{cis} \frac{\pi}{3} = 1 + i\sqrt{3} \\ \bullet\ 2 \operatorname{cis} \frac{5\pi}{3} = 1 - i\sqrt{3} \end{array}}$$