Differential Equations

Ex $\quad t^2 \frac{dy}{dt} = t+1, \; t>0$ ← To solve means to find the function y

$$\frac{dy}{dt} = \frac{t+1}{t^2}$$

$$\int 1\,dy = \int \frac{t+1}{t^2}\,dt$$ ← separation of variables step

$$y = \int \frac{t}{t^2}\,dt + \int \frac{1}{t^2}\,dt$$

$$y = \int \frac{1}{t}\,dt + \int t^{-2}\,dt$$

$$y = \ln t + \frac{1}{-1} t^{-1} + C$$

$$\boxed{y = \ln t - \frac{1}{t} + C}$$ general solution

Find the particular solution for which $y(e^2) = 4$

$$\boxed{y = \ln t - \frac{1}{t} + c} \quad \text{general solution}$$

$$4 = \ln e^2 - \frac{1}{e^2} + C$$

vertical shift

$$4 = 2 - \frac{1}{e^2} + C$$

$$C = 2 + \frac{1}{e^2}$$

particular solution through $(e^2, 4)$ is

$$\boxed{y = \ln t - \frac{1}{t} + 2 + \frac{1}{e^2}}$$

Ex. Find the particular solution

for $\frac{dy}{dt} = yt(t^2+1)^3$

such that $y = 2$ when $t = 1$.

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separation step: $\int \frac{1}{y}\, dy = \frac{1}{2} \cdot \int 2t\,(t^2+1)^3\, dt$

inner function $\rightarrow$ $u = t^2 + 1$

$du = 2t\, dt$

$$\ln|y| = \frac{1}{2}\int u^3\, du$$

$$\ln|y| = \frac{1}{2}\cdot\frac{1}{4}u^4 + C$$

$$\ln|y| = \frac{1}{8}(t^2+1)^4 + C \leftarrow \text{general solution}$$

$$y = e^{\frac{1}{8}(t^2+1)^4 + C}$$

$$e^{\frac{1}{8}(t^2+1)^4} \cdot \cancel{e^C} A$$

$$x^{m+n} = x^m \cdot x^n$$

$$2 = e^{2+C} = e^2 \cdot \cancel{e^C} A \Rightarrow A = \frac{2}{e^2}$$

particular solution: $\boxed{y = \frac{2}{e^2} e^{\frac{1}{8}(t^2+1)^4}}$

Logistic Growth

Ex. $$\frac{dP}{dt} = kP(M-P), \quad k \text{ and } M \text{ are constants, } t \geq 0, \; P \geq 0$$

$$\underbrace{\int \frac{dP}{P(M-P)}}_{\text{use partial fractions}} = k\int dt \quad \star$$

$$\frac{1}{P(M-P)} = \frac{A}{P} + \frac{B}{M-P}$$

Find A + B

$$1 = A(M-P) + BP$$

Let $P=0$: $\quad 1 = AM + 0 \rightarrow A = \frac{1}{M}$

Let $P=M$: $\quad 1 = 0 + BM \rightarrow B = \frac{1}{M}$

$$\int \frac{1}{P(M-P)}\,dP = \frac{1}{M}\int \frac{1}{P}\,dP - \frac{1}{M}\int \frac{-1}{M-P}\,dP$$

$$= \frac{1}{M}\left(\ln P - \ln(M-P)\right)$$

$u = M-P$

$du = -1\,dP$

$\int \frac{du}{u}$

$$\frac{1}{M}\ln\frac{P}{M-P} = kt + C$$

Now, solve for P (the indep. variable)

$$\ln\frac{P}{M-P} = kMt + A$$

$$\frac{P}{\cancel{M-P}} = \left(e^{kMt}\cdot \cancel{e^{A}}\, B\right)(M-P)$$

$$P = M\left(Be^{kMt}\right) - \underbrace{P\left(Be^{kMt}\right)}$$

$$P\left(1 + Be^{kMt}\right) = MBe^{kMt}$$

$$\boxed{P(t) = \frac{MBe^{kMt}}{1 + Be^{kMt}}}$$