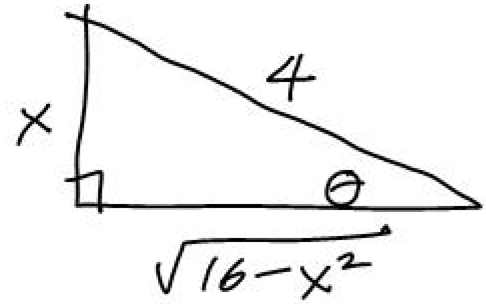
Trigonometric Substitution

(a) $\int \sqrt{16-x^2}\,dx$



$$\frac{x}{4} = \sin\theta$$

$$x = 4\sin\theta$$

$$dx = 4\cos\theta\,d\theta$$

$$= \int \sqrt{16 - 16\sin^2\theta}\; 4\cos\theta\,d\theta$$

$$= \int 4\sqrt{1-\sin^2\theta} \cdot 4\cos\theta\,d\theta$$

$$= 16\int \cos\theta \cdot \cos\theta\,d\theta$$

$$= 16\int \cos^2\theta\,d\theta = 16\left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right) + C$$

$$= 8\theta + 4\sin 2\theta + C = 8\theta + 4\cdot 2\sin\theta\cos\theta + C$$

$$= 8\sin^{-1}\left(\frac{x}{4}\right) + 8\cdot\frac{x}{4}\cdot\frac{\sqrt{16-x^2}}{4} + C$$

$$= 8\sin^{-1}\left(\frac{x}{4}\right) + \frac{x\sqrt{16-x^2}}{2} + C$$

(b) $-\frac{1}{2}\int -2x\sqrt{16-x^2}\,dx = -\frac{1}{2}\int u^{1/2}\,du$

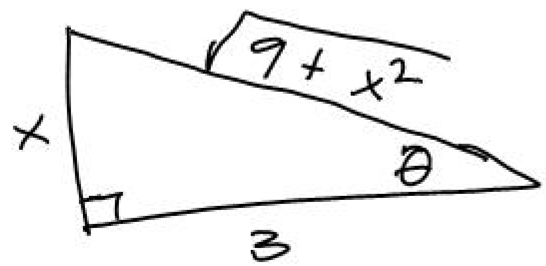
$u = 16 - x^2$

$du = -2x\,dx$

$= -\frac{1}{3}u^{3/2} + C$

$= -\frac{1}{3}\left(16-x^2\right)^{3/2} + C$

(c) $\int \frac{dx}{\sqrt{9+x^2}}$



$= \int \frac{3\sec^2\theta\,d\theta}{\sqrt{9+9\tan^2\theta}}$

$\frac{x}{3} = \tan\theta$

$x = 3\tan\theta$

$dx = 3\sec^2\theta\,d\theta$

$= \int \frac{\cancel{3}\sec^2\theta\,d\theta}{\cancel{3}\sqrt{1+\tan^2\theta}}$

$= \int \frac{\sec^2\theta}{\sec\theta}\,d\theta = \int \sec\theta\,d\theta$

$= \ln\left|\sec\theta + \tan\theta\right| + C_1$

$= \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + C_1$

$= \ln\left|\sqrt{9+x^2} + x\right| + C$

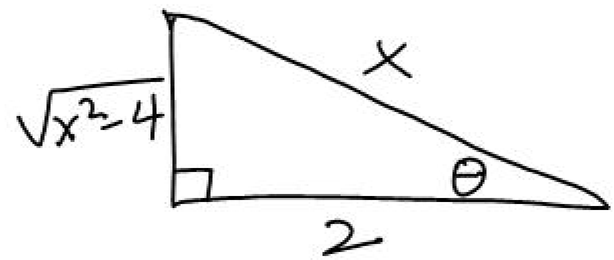
(d) $\displaystyle\int \frac{2x}{\sqrt{9+x^2}}\,dx = \int u^{-1/2}\,du = 2u^{1/2}+C$

$u = 9+x^2$

$du = 2x\,dx$

$= 2\sqrt{9+x^2}+C$

(e) $\displaystyle\int \frac{\sqrt{x^2-4}}{x}\,dx$



$$= \int \frac{\sqrt{4\sec^2\theta - 4}}{2\cancel{\sec\theta}}\, 2\cancel{\sec\theta}\tan\theta\,d\theta$$

$\frac{x}{2} = \sec\theta$

$x = 2\sec\theta$

$dx = 2\sec\theta\tan\theta\,d\theta$

$$= \int \frac{\cancel{2}\sqrt{\sec^2\theta - 1}}{\cancel{2}} \cdot 2\cdot\tan\theta\,d\theta$$

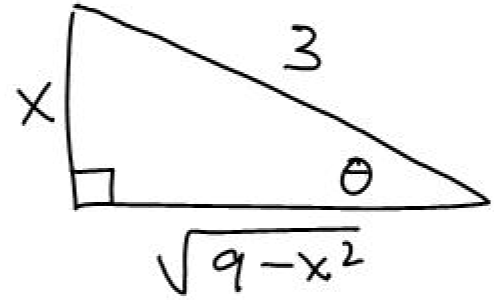
$$= 2\int \tan^2\theta\,d\theta = 2\int(\sec^2\theta - 1)\,d\theta$$

$$= 2\left(\tan\theta - \theta\right) + C$$

$$= 2\left(\frac{\sqrt{x^2-4}}{2} - \sec^{-1}\left(\frac{x}{2}\right)\right) + C$$

$$= \sqrt{x^2-4} - 2\sec^{-1}\left(\frac{x}{2}\right) + C$$

(f) $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx$



$\frac{x}{3} = \sin\theta$

$x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

$$= \int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta}\, 3\cos\theta\, d\theta$$

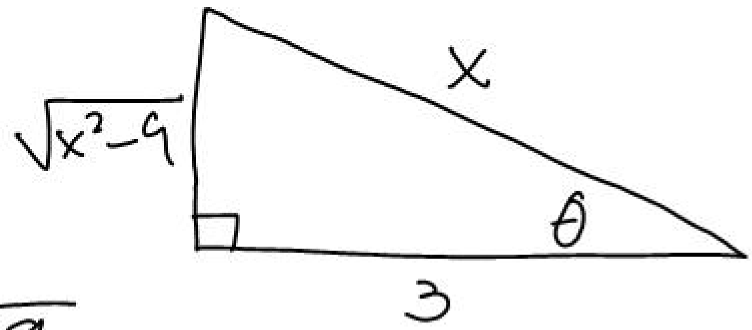
$$= \int \frac{\cancel{3}\sqrt{1-\sin^2\theta}}{\cancel{9}\sin^2\theta} \cdot \cancel{3}\cos\theta\, d\theta$$

$$= \int \frac{\cos\theta}{\sin^2\theta}\cos\theta\, d\theta = \int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta = \int \cot^2\theta\, d\theta$$

$$= \int (\csc^2\theta - 1)\, d\theta = -\cot\theta - \theta + C$$

$$= -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$$

9) $$\int \frac{dx}{x^3\sqrt{x^2-9}}$$



$$= \int \frac{\cancel{3}\,\cancel{\sec\theta}\,\tan\theta\, d\theta}{9\,\cancel{27}\,\sec^{\cancel{3}\,2}\theta\,\sqrt{9\sec^2\theta-9}}$$

$$\frac{x}{3} = \sec\theta$$

$$x = 3\sec\theta$$

$$dx = 3\sec\theta\tan\theta\, d\theta$$

$$= \int \frac{\tan\theta\, d\theta}{9\sec^2\theta\cdot 3\sqrt{\sec^2\theta-1}}$$

$$= \int \frac{\cancel{\tan\theta}\, d\theta}{9\cdot\sec^2\theta\cdot 3\cdot\cancel{\tan\theta}}$$

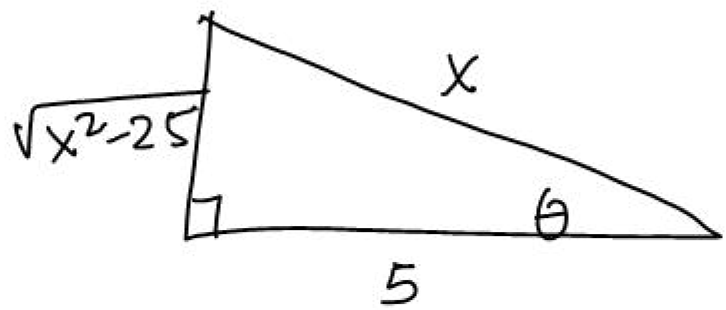
$$= \frac{1}{27}\int \cos^2\theta\, d\theta = \frac{1}{27}\left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right) + C$$

$$= \frac{1}{54}\theta + \frac{1}{108}\cdot 2\sin\theta\cos\theta + C$$

$$= \frac{1}{54}\sec^{-1}\left(\frac{x}{3}\right) + \frac{1}{54}\cdot\frac{\sqrt{x^2-9}}{x}\cdot\frac{3}{x} + C$$

$$= \frac{1}{54}\sec^{-1}\left(\frac{x}{3}\right) + \frac{\sqrt{x^2-9}}{18x^2} + C$$

(h) $\displaystyle\int \frac{dx}{\sqrt{x^2-25}}$



$$= \int \frac{5\sec\theta\tan\theta\, d\theta}{\sqrt{25\sec^2\theta - 25}}$$

$$\frac{x}{5} = \sec\theta$$

$$x = 5\sec\theta$$

$$dx = 5\sec\theta\tan\theta\, d\theta$$

$$= \int \frac{\cancel{5}\sec\theta\tan\theta}{\cancel{5}\sqrt{\sec^2\theta - 1}}\, d\theta$$

$$= \int \frac{\sec\theta\cancel{\tan\theta}}{\cancel{\tan\theta}}\, d\theta = \int \sec\theta\, d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + C_1$$

$$= \ln\left|\frac{x}{5} + \frac{\sqrt{x^2-25}}{5}\right| + C_1$$

$$= \ln\left|x + \sqrt{x^2-25}\right| + C$$

(i) $\frac{1}{2}\int \frac{2x}{\sqrt{x^2-25}}\,dx = \frac{1}{2}\int u^{-1/2}\,du$

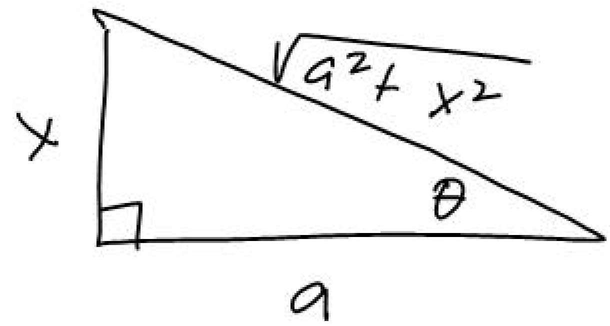
$u = x^2 - 25$

$du = 2x\,dx$

$= u^{1/2} + C$

$= \sqrt{x^2-25} + C$

(j) $\int \frac{dx}{a^2+x^2}$



$\frac{x}{a} = \tan\theta$

$x = a\cdot\tan\theta$

$dx = a\cdot\sec^2\theta\,d\theta$

$= \int \frac{a\cdot\sec^2\theta\,d\theta}{a^2 + a^2\tan^2\theta}$

$= \int \frac{\cancel{a}\,\sec^2\theta\,d\theta}{a^{\cancel{2}}(1+\tan^2\theta)}$

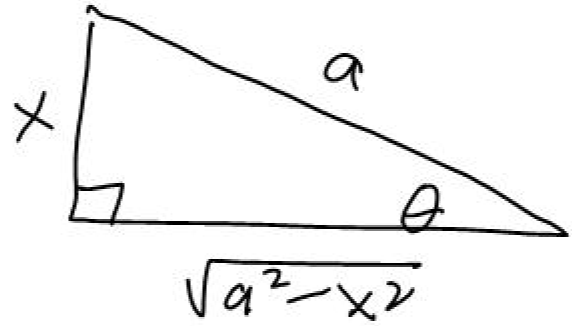
$= \frac{1}{a}\int \frac{\sec^2\theta}{\sec^2\theta}\,d\theta$

$= \frac{1}{a}\int d\theta = \frac{1}{a}\cdot\theta + C = \frac{1}{a}\tan^{-1}\theta + C$

Memorize

$$\boxed{\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C}$$

(k) $\int \frac{dx}{\sqrt{a^2-x^2}}$



$\frac{x}{a} = \sin\theta$

$x = a \cdot \sin\theta$

$dx = a \cdot \cos\theta \, d\theta$

$= \int \frac{a \cos\theta \, d\theta}{\sqrt{a^2 - a^2 \sin^2\theta}}$

$= \int \frac{\cancel{a} \cdot \cos\theta \, d\theta}{\cancel{a} \sqrt{1-\sin^2\theta}}$

$= \int \frac{\cos\theta}{\cos\theta} d\theta$

$= \int d\theta = \sin^{-1}\left(\frac{x}{a}\right) + C$

Memorize

$$\boxed{\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C}$$