u-substitution

(A way to get antiderivatives for composite functions, sometimes)

Ex. $\frac{1}{9}\int 9x^2 \cdot \overbrace{(3x^3+2)^4}^{\text{composite}}\,dx = \frac{1}{9}\int u^4\,du$

inner function $\rightsquigarrow$ $u = 3x^3+2$

$\frac{du}{\cancel{dx}} = 9x^2\,dx$

$= \frac{1}{9}\cdot\frac{1}{5}u^5 + C$

$= \frac{1}{45}(3x^3+2)^5 + C$

The Power Rule for Antiderivatives

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$$

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$$

check

use the chain rule

$$\frac{d}{dx}\left[\frac{1}{45}(3x^3+2)^5+C\right] = \frac{1\cancel{5}}{\cancel{9}\cancel{45}}(3x^3+2)^4 \cdot \cancel{9}x^2$$

$$= x^2(3x^3+2)^4$$

Ex. $\frac{1}{2}\int \frac{2x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du$

$u = x^2+1$

$du = 2x\,dx$

$$= \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln|x^2+1| + C$$

$$= \frac{1}{2}\ln(x^2+1) + C$$

check

$$\frac{d}{dx}\left[\frac{1}{2}\ln(x^2+1)\right] = \frac{1}{\cancel{2}} \cdot \frac{1}{x^2+1} \cdot \cancel{2}x$$

$$= \frac{x}{x^2+1}$$

Bad news

$$\int \sqrt{x^3+1}\, dx$$

$u = x^3 + 1$

$du = \underline{3x^2}\, dx$

u-sub. fails

Ex. $\displaystyle\int x\sqrt{x+1}\, dx = \int (u-1)\, u^{1/2}\, du$

$u = x+1 \rightsquigarrow \underline{x = u-1}$

$du = \underline{dx}$

$$= \int \left(u^{3/2} - u^{1/2} \right) du$$

$$= \frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} + C$$

$$= \frac{2}{5} (x+1)^{5/2} - \frac{2}{3} (x+1)^{3/2} + C$$

Ex $-\int \frac{-\sin x}{\cos x}\,dx = -\int \frac{du}{u} = -\int \frac{1}{u}\,du$

$u = \cos x$

$du = -\sin x\,dx$

$-\ln|u| + C$

$\underline{-\ln|\cos x| + C}$

$$\int \tan x\,dx = -\ln|\cos x| + C$$

$$= \ln|\cos x|^{-1} + C$$

$$= \ln|\sec x| + C$$