Maclaurin Polynomials

(centered at $x=0$) n is the index

$$P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n$$

$$= f(0) + f'(0)\cdot x + \frac{f''(0)}{2!} x^2$$

$$+ \frac{f'''(0)}{3!} x^3 + \cdots + \frac{f^{(N)}(0)}{N!} x^N$$

Maclaurin polynomials to know and love:

$$\sin x = \overset{0}{x} - \overset{1}{\frac{x^3}{3!}} + \overset{2}{\frac{x^5}{5!}} \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad x \in \mathbb{R}$$

$$(2n+1)! = (2n+1)(2n)(2n-1)(2n-2)\cdots 1$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad x \in \mathbb{R}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

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Ex. The Euler number:

$$e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

Taylor Polynomials
(centered at $x=c$)

$$P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

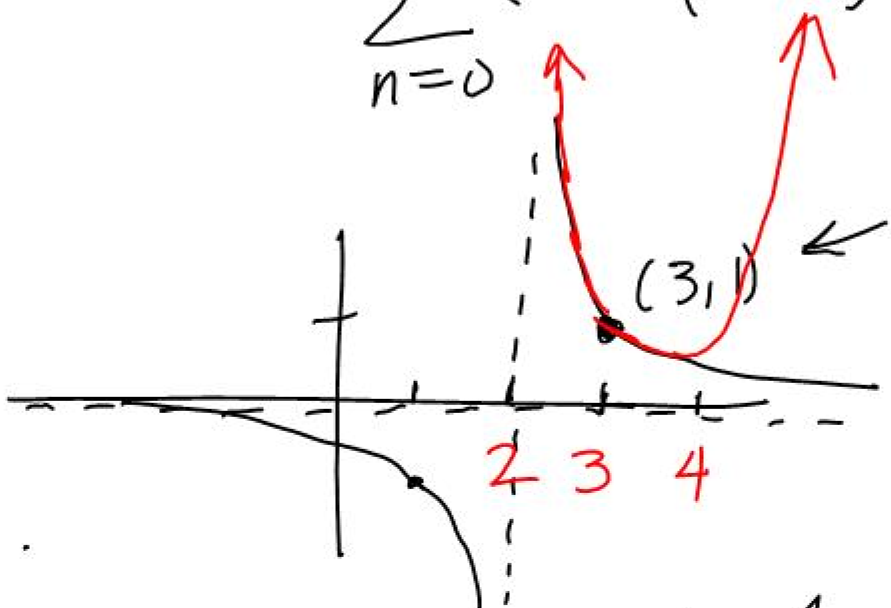
Ex. Write the Taylor polynomial for

$$f(x) = \frac{1}{x-2} \text{ centered at } x=3$$

n	$f^{(n)}(x)$	$f^{(n)}(3)$	$\frac{f^{(n)}(3)}{n!}$
0	$(x-2)^{-1}$	$1 \div 0!$	1
1	$-(x-2)^{-2}$	$-1 \div 1!$	$-1(x-3)$
2	$2(x-2)^{-3}$	$2 \div 2!$	$1(x-3)^2$
3	$-6(x-2)^{-4}$	$-6 \div 3!$	-1
4	$24(x-2)^{-5}$	24	1

$$\frac{1}{x-2} = 1 - (x-3) + (x-3)^2 - (x-3)^3 \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n (x-3)^n, \quad 2 < x < 4$$



radius of convergence is 1.

$\boxed{\text{(2, 4) is the interval of convergence}}$

Finding the interval of convergence

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rho $\rho\ \rho\ \rho$

<u>The Ratio Test</u>

For $\displaystyle\sum_{n=0}^{\infty} a_n$, Let $\displaystyle\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

If $\rho < 1$, the series converges absolutely.

If $\rho > 1$, the series diverges.

If $\rho = 1$, the test fails.
(use another test)

Ex. Find the radius of convergence for $\displaystyle\sum_{n=0}^{\infty} (-1)^n (x-3)^n$

$$\rho = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-3)^{n+1}}{(-1)^n (x-3)^n} \right| = \cancel{\lim_{n\to\infty}} |x-3| < 1$$

$-1 < x-3 < 1$

$2 < x < 4$

$$\boxed{R = 1}$$

Find the radius of convergence

for $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ $\qquad 2(n+1)$

$$\rho = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} \frac{x^{2n+2}}{(2n+2)!}}{(-1)^n \frac{x^{2n}}{(2n)!}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^2}{(2n+2)(2n+1)} \right| = 0 < 1$$

$\rho < 1$ for all x-values

$(2n+2)! = (2n+2)(2n+1)(2n)!$

converges for $-\infty < x < \infty$

$$\boxed{R = \infty}$$

Ex. Write P_{13} for $f(x) = x \cdot \cos x^3$ centered at 0.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots$$

$$\cos x^3 = 1 - \frac{(x^3)^2}{2!} + \frac{(x^3)^4}{4!} \cdots$$

$$1 \qquad 0 \qquad 1 \qquad 2$$

$$\boxed{x \cdot \cos x^3 \approx x - \frac{x^7}{2!} + \frac{x^{13}}{4!}}$$

$3(2n)+1$

$$\boxed{x \cdot \cos x^3 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+1}}{(2n)!}}$$