Polynomial Approximations

3^{rd} degree polynomial approximation

$$\boxed{P_3(x) = \overset{0}{f(c)} + \overset{1}{f'(c)}(x-c) + \underset{2}{\frac{f''(c)}{2!}}(x-c)^2 + \underset{3}{\frac{f'''(c)}{3!}}(x-c)^3}$$

Ex. Write $P_3(x)$ for $f(x) = \sin x$ centered at $x = 0$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$\frac{f^{(n)}(0)}{n!}$
0	$\sin x$	$0 \quad \div 0!$	0
1	$\cos x$	$1 \quad \div 1!$	1
2	$-\sin x$	$0 \quad \div 2!$	0
3	$-\cos x$	$-1 \quad \div 3!$	$-\frac{1}{3!}$

$$\sin x \approx x - \frac{1}{3!}x^3$$

sine is an odd function

$$P_{\cancel{5}7}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$n = \quad 0 \qquad 1 \qquad 2 \qquad 3$

$$\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin x}$$

Write $P_8(x)$ for $\cos x$ centered at $x = 0$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$\frac{f^{(n)}(0)}{n!}$
0	$\cos x$	1	1
1	$-\sin x$	0	0
2	$-\cos x$	-1	$-1/2!$
3	$\sin x$	0	0
4	$\cos x$	$+1$	$1/4!$
5	$-\sin x$	0	0
6	$-\cos x$	-1	$-1/6!$
7	$\sin x$	0	0
8	$\cos x$	1	$1/8!$

$$\boxed{\cos x \approx \overset{0}{1} - \overset{1}{\frac{x^2}{2!}} + \overset{2}{\frac{x^4}{4!}} - \overset{3}{\frac{x^6}{6!}} + \frac{x^8}{8!}}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Write $P_5(x)$ for e^x centered at $x=0$. This is the Maclaurin Polynomial for e^x.

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$\frac{f^{(n)}(0)}{n!}$	const.
0	e^x	1	1	
1	e^x	1	1	x
2	e^x	1	$1/2!$	x^2
3	e^x	$1 \div 3!$	$1/3!$	x^3
4	e^x	1	$1/4!$	x^4
5	e^x	1	$1/5!$	x^5

$$e^x \approx \overset{0}{1} + \overset{1}{x} + \overset{2}{\frac{x^2}{2!}} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\frac{f^{(n)}(c)}{n!}(x-c)^n$$

Write $P_{10}(x)$ for $\sin x^2$ at $x=0$.

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots$$

$$\sin x^2 \approx x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!}$$

$$= x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!}$$

Write $P_5(x)$ for $f(x) = \frac{e^{x^2}}{x}$ at $x=0$

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots$$

$$e^{x^2} \approx 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} \cdots$$

$$\boxed{\frac{e^{x^2}}{x} \approx \frac{1}{x} + x + \frac{x^3}{2!} + \frac{x^5}{3!}}$$