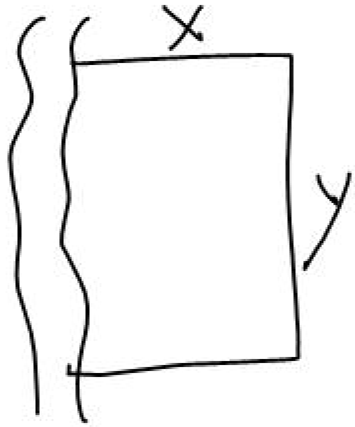
Ex.



(a) A $10000\ ft^2$ corral is to be built as shown. Find x so that the least amount of fence is used.

(b) You have 500 m of fence. Find the largest possible area.

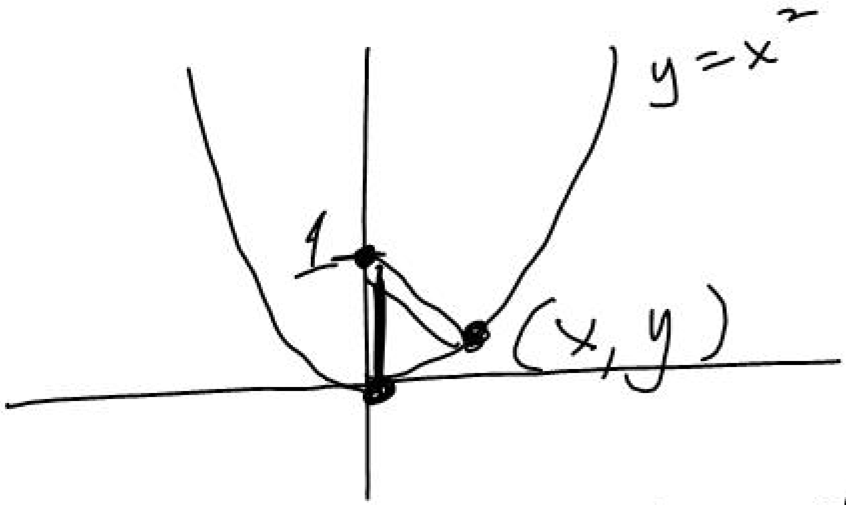
$$2x + y = 500$$
$$y = 500 - 2x$$

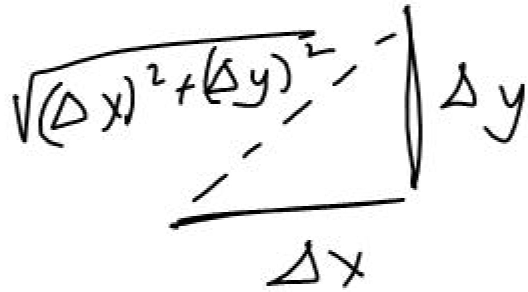
$$A = x(500 - 2x)$$
$$= 500x - 2x^2$$

$$A' = 500 - 4x = 0$$
$$x = 125\,m$$

$$\text{Area} = 125(500 - 250)$$
$$= 125(250)$$

$$\begin{array}{r} 125 \\ \times\ 250 \\ \hline 6250 \\ 250 \\ \hline 31250\ m^2 \end{array}$$





Find the point on the parabola closest to $(0,1)$ in the 1^{st} quadrant.

$$d = \sqrt{(x-0)^2 + (y-1)^2}$$

$y = x^2$

$$d = \sqrt{x^2 + (x^2-1)^2}$$

$$d = \sqrt{x^4 - x^2 + 1} = (x^4 - x^2 + 1)^{1/2}$$

$$d' = \cancel{\tfrac{1}{2}(x^4 - x^2 + 1)^{-1/2}}\,(4x^3 - 2x) = 0$$

$$2x(2x^2 - 1) = 0$$

$\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$

$(0, 0)$ 1 $\quad x = 0 \quad x = \frac{1}{\sqrt{2}}$

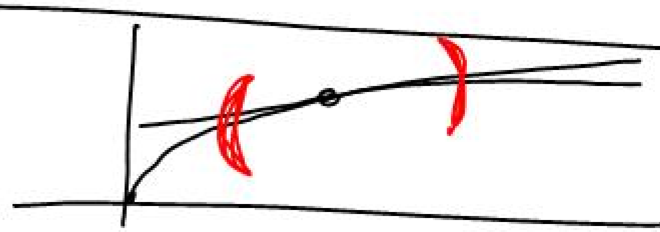
$(0,1)$ $\boxed{\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)}$

$$\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{4}} = \frac{\sqrt{3}}{2}$$

$$\approx 0.866$$

L'Hôpital, Optimization, Related Rates

Tangent Lines



Write an equation for the line tangent to $y = \sqrt{x}$ at $x = 4$.

point $(4,2)$

slope $\frac{1}{4}$

$$y' = \frac{1}{2} x^{-1/2}$$

$$y'\Big|_{x=4} = \frac{1}{2} \cdot 4^{-1/2} = \frac{1}{4}$$

Tangent Line

$$y - 2 = \frac{1}{4}(x - 4)$$

$$P_1(x) \longrightarrow \boxed{y = 2 + \frac{1}{4}(x - 4)}$$

$\uparrow$

1^{st} degree polynomial

Use the tangent line to approximate

$f(4.1) = \sqrt{4.1} \approx 2.024845$

$P_1(4.1) = 2 + \frac{1}{4}(4.1 - 4)$

$.25$

$= 2 + \frac{1}{4}(0.1)$

$= 2 + 0.025$

$= 2.025$ ← % error

$$\frac{|2.025 - \sqrt{4.1}|}{\sqrt{4.1}} \times 100$$

$= \underline{0.008\%}$

Tangent Parabolas

$$\text{Line: } P_1 = \underset{f(4)}{2} + \underset{f'(4)}{\tfrac{1}{4}}\left(x - \overset{\text{center (point of tangency)}}{4}\right)$$

$$\text{Parabola: } P_2 = 2 + \tfrac{1}{4}(x-4) + \underset{\frac{f''(4)}{2}}{\boxed{\tfrac{-1}{64}}}(x-4)^2$$

$f(x) = x^{1/2}$	$f''(4) = \frac{-1}{4}(4)^{-\frac{3}{2}}$
$f'(x) = \frac{1}{2}x^{-1/2}$	$= -\frac{1}{4}\cdot\frac{1}{8}$
$f''(x) = -\frac{1}{4}x^{-3/2}$	$= -\frac{1}{32}$

$$P_2 = a_0 + a_1(x-c) + a_2(x-c)^2$$

$$\boxed{a_0 = f(c)}$$

$$P_2' = a_1 + 2a_2(x-c) =$$

$$P_2'(c) = \boxed{a_1 = f'(c)}$$

$$P_2'' = 2a_2 =$$

$$P_2''(c) = 2a_2 = f''(c)$$

$$\boxed{a_2 = \frac{f''(c)}{2}}$$

Third-degree Polynomial Approx

$$P_3(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3$$

$$a_0 = \frac{f(c)}{0!} \qquad a_2 = \frac{f''(c)}{2!}$$

$$a_1 = \frac{f'(c)}{1!} \qquad a_3 = \frac{f'''(c)}{3!}$$

$$P_3'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2$$

$$P_3''(x) = 2a_2 + 3!\,a_3(x-c)$$

$$P_3'''(x) = 3!\,a_3$$

$$P_3'''(c) = 3!\,a_3 = f'''(c)$$

$$\boxed{a_3 = \frac{f'''(c)}{3!}}$$

Write P_3 for $f(x) = \sqrt{x}$

at $x = 4$.

$$\boxed{P_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3}$$

$$f'''(x) = \frac{3}{8}x^{-5/2}$$

$$f'''(4) = \frac{3}{8} \cdot \frac{1}{32} = \frac{3}{256}$$

$$\frac{\cancel{3}}{256} \cdot \frac{1}{\cancel{6}\,2}$$